

QCD Evolution by Finite Element Methods

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Abstract

A simple, new method for solving for the Q^2 evolution of parton distributions in perturbative QCD using cubic splines is described and applied to the evolution of nonsinglet quark distributions.

1 Introduction

The efficacy of using perturbative QCD to describe processes at large momentum transfer in terms of process independent parton distributions has spawned an industry devoted to the determination of these distributions[1]. Central to this endeavour is the ability to solve the Altarelli-Parisi equations[2], which govern the Q^2 evolution of parton distributions, so that data taken at different energies can be used to constrain the form of the parton distributions. The evolution equations have been solved in a variety of ways, ranging from brute force integration of the integro-differential equations[3], to orthogonal polynomial expansions[4], to solutions in terms of Bernstine polynomials for next to leading order evolution[5]. In this report, a straightforward new method for solving the Altarelli-Parisi equations in terms of a small number of cubic splines is described and illustrated for the case of nonsinglet quark distributions. Since the method relies only on the smoothness of the parton distributions and the moment structure of the QCD evolution equations, it is applicable to singlet and nonsinglet distributions in both leading and next to leading order with equal ease.

2 The Method

In this section, the application of finite element methods to the problem of QCD evolution is demonstrated for the simple case of a nonsinglet quark distribution. Given a nonsinglet quark distribution $q_{NS}(x, Q_0^2)$ at some initial renormalization scale Q_0^2 , we are looking for a new distribution evolved to a different Q^2 according to QCD perturbation theory. To leading order, this requires

$$\mathcal{M}_n(Q^2) = \mathcal{M}_n(Q_0^2) \times \left(\frac{\log(Q_0^2)}{\log(Q^2)} \right)^{d_n^{NS}}, \quad (1)$$

where $\mathcal{M}_n(Q^2) = \int_0^1 dx x^{n-1} q_{NS}(x, Q^2)$ is the n th moment of the quark distribution, and d_n^{NS} is the anomalous dimension of the n th nonsinglet, twist two operator.

In principle, knowledge of all the moments is required to uniquely determine $q_{NS}(x, Q^2)$, but the distributions used in QCD phenomenology are peaked at small values of x , and all but a few of the moments are small. Exploiting this fact, we proceed by numerically calculating the first N moments of the desired distribution in terms of a set of N suitably chosen integration points and weights. Thus,

$$\mathcal{M}_n(Q^2) \approx \sum_{i=1}^N x_i^{n-1} w_i q_{NS}(x_i, Q^2), \quad (2)$$

where $\{x_i\}$ are the integration points and $\{w_i\}$ are the weights. Assuming that the integrals are well approximated, this yields a matrix equation relating the first N moments of the distribution to the values of the distribution

function at the points $\{x_i\}$. In principle, the distribution can be solved for by taking the inverse of the matrix and the limit as the number of integration points gets large. As a practical method, this procedure fails since the matrix,

$$A_{ni} = x_i^{n-1} w_i, \quad (3)$$

is difficult to invert numerically when N is large. For relatively small values of N , however, A_{ni} can be inverted easily and quickly by any one of a variety of standard techniques. The result is an approximation of the values of the parton distribution at N points. All that is left is to fill in the rest of the curve between these points. Since the distribution is expected to be smooth and non-oscillatory, the remainder of the curve is found by interpolating between the points x_i using cubic splines[6]. Since the functions of interest are singular at $x = 0$, some care must be taken when choosing the integration scheme and spline basis so that the function may be adequately fit at small x . While other techniques for dealing with this problem exist in the literature[6], here we simply change the variable in the numerical integration and spline fit to $y = x^\alpha$, with $\alpha < 1$, so that small values of x have increased weight in the integrations and the curve is better approximated by a polynomial in y near $x = 0$.

3 Results and Discussion

The essence of the method outlined in the previous section is the reconstruction of an essentially arbitrary curve from its moments. The simplest test available is to decompose a curve into moments and then reconstruct it without any change in Q^2 . In Figure 1, the results are displayed for a curve typical of those encountered in QCD phenomenology, $xq_{NS}(x) = \sqrt{x}(1-x)^3$, using a Gauss-Legendre integration scheme with $\alpha = 2/3$, $N=3,5,7$ or 10, and splines that are functions of $y = \sqrt{x}$. For $N=3$ the procedure has clearly failed to reproduce the original, while for $N=5$ the reconstructed curve deviates only slightly. For $N=7$ and 10, however, the reconstructed curves are virtually indistinguishable from the original. In Figure 2, a set of 10 splines is used to perform the leading order evolution of the valence d quark distribution parametrized by Morfin and Tung(MT)[3] from an initial scale $Q_0^2 = 4 \text{ GeV}^2$ to $Q^2 = 10$ and 50 GeV^2 . Using $\alpha = 1/2$ for both the integration points and splines, the MT curves are well reproduced. (The small deviation from the MT curve is a reflection of the approximate nature of the parametrization of the Q^2 dependence of $xd_V(x)$ in reference 4.)

The finite element method for solving the Altarelli-Parisi equations has a number of conceptual and practical advantages. To begin, the method is conceptually straightforward, relying only on the moment structure of the evolution equations and the smoothness of the quark distributions. Since these properties hold for all parton distributions the method is applicable,

essentially without change, to both singlet and nonsinglet distributions in both leading and next to leading order in α_s . In addition, one need not commit to a particular form for the parton distributions when fitting data, but can instead treat the moments of the distribution, which are related to the physically relevant twist two matrix elements, as fit parameters. Practically, the spline methods used here are more stable than schemes involving orthogonal polynomials, which are prone to global oscillations when singularities are encountered. Since the elements of the scheme we have described, numerical integration, matrix inversion, and spline interpolation, are extensively used in other applications, the method is easily implemented. If greater accuracy is desired, the method can be altered to include any non-integral, finite moment of the parton distributions by simply generating the appropriate anomalous dimensions and coefficients. Since the matrix A_{ni} need only be inverted once to fit data over a large range in Q^2 , the current method is faster than brute force integration of the Altarelli-Parisi equations.

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Figure Captions

- Figure 1 Reconstruction of the curve $\sqrt{x}(1-x)^3$ using 3,5,7 and 10 splines.
- Figure 2 Leading order evolution of the valence d quark distribution of Morfin and Tung(MT)[3].

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